Transfinite Barycentric Coordinates for Arbitrary Planar Domains

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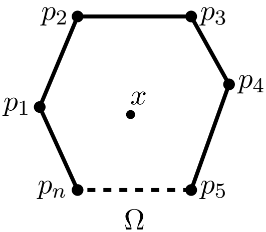
Generalized barycentric coordinates

Given a polygon Ω with n vertices $p_1, p_2, \ldots, p_n$, and any $x \in \Omega$, find coordinates $\boldsymbol{\lambda}(x) = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ such that

$$x = \sum_{i=1}^{n} \lambda_i(x) p_i, \quad \sum_{i=1}^{n} \lambda_i(x) = 1, \quad \lambda_i(v_j) = \delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$



- $\boldsymbol{\lambda}$ are *generalized barycentric coordinates* of x w.r.t. Ω.

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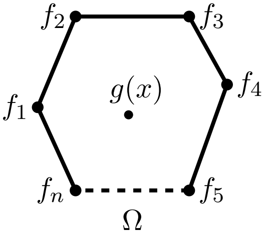
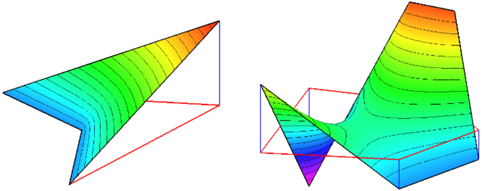


- $\boldsymbol{\lambda}$ are *generalized barycentric coordinates* of x w.r.t. Ω.

Considering data $f_1, f_2, \ldots, f_n$ corresponding to the vertices, we can interpolate the data at any point $x \in \Omega$ by

$$g(x) = \sum_{i=1}^{n} \lambda_i(x) f_i, \quad \sum_{i=1}^{n} \lambda_i(x) = 1$$

(Generalized barycentric interpolation)



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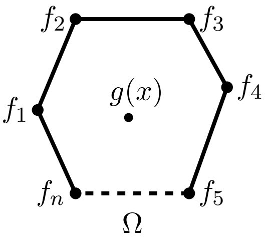

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(Generalized barycentric interpolation)

- When $n > 3$, such $\boldsymbol{\lambda}$ are usually not unique.
- Looking forward to finding $\boldsymbol{\lambda}$ that satisfy some properties.

Applications

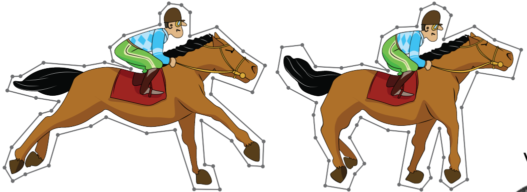
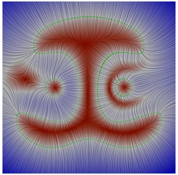
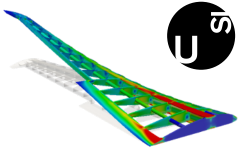
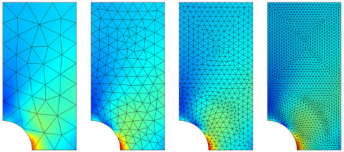


image warping

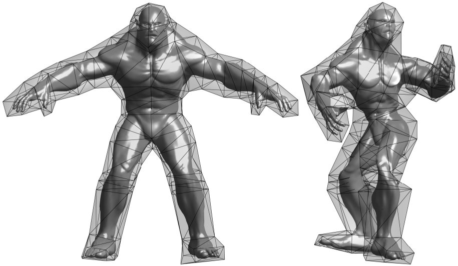


vector field interpolation

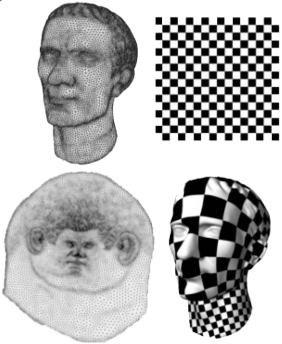




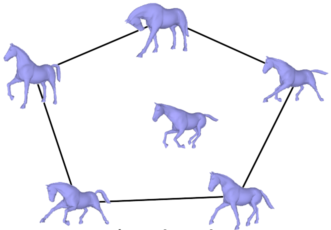
finite element method



mesh deformation

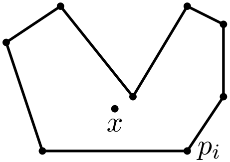


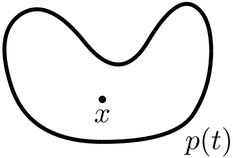
mesh parameterization



mesh animation

Transfinite barycentric coordinates





Polygon: $\{p_i \colon i = 1, \ldots, n\}$

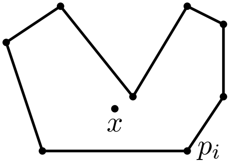
Curve: $\{p(t) \colon t \in [a, b]\}$, $p(a) = p(b)$

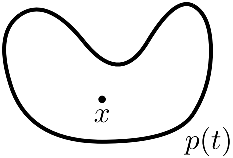
generalized barycentric coordinates $\lambda_i(x)$

transfinite barycentric kernel $\lambda(x, t)$

$\sum_{i=1}^{n} \lambda_i(x) p_i = x$	linear reproduction	$\int_a^b \lambda(x, t) p(t) \, \mathrm{d}t = x$
$\sum_{i=1}^{n} \lambda_i(x) = 1$	partition of unity	$\int_a^b \lambda(x, t) \, \mathrm{d}t = 1$
$\lambda_i(p_j) = \delta_{i,j}$	Lagrange property	$\lambda(p(s), t) = \delta(s - t)$

Transfinite barycentric coordinates





Polygon: $\{p_i : i = 1, \ldots, n\}$

generalized barycentric coordinates $\lambda_i(x)$

$$\sum_{i=1}^{n} \mu_i(x)(p_i - x) = 0$$

$$\lambda_i(x) = \mu_i(x) \Big/ \sum_{j=1}^{n} \mu_j(x)$$

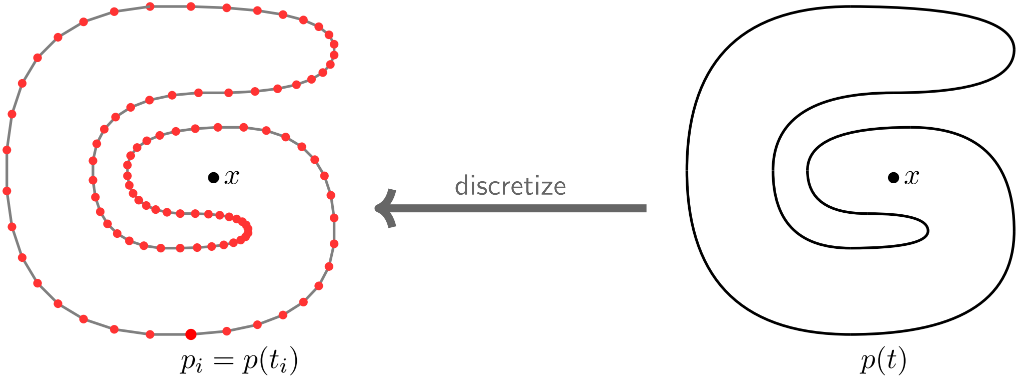
Curve: $\{p(t) : t \in [a, b]\}$, $p(a) = p(b)$

transfinite barycentric kernel $\lambda(x, t)$

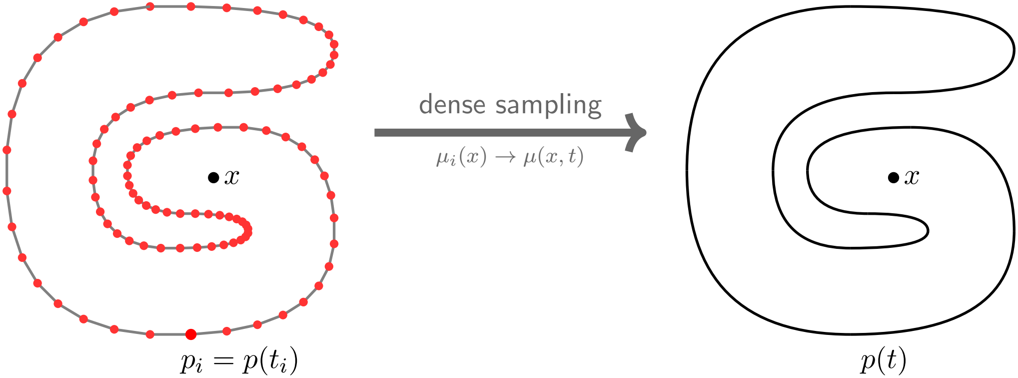
$$\int_a^b \mu(x, t)(p(t) - x)\,\mathrm{d}t = 0$$

$$\lambda(x, t) = \mu(x, t) \Big/ \int_a^b \mu(x, s)\,\mathrm{d}s$$

Discretize the continuous boundary



Discretize the continuous boundary



Related work

- Wachspress kernel [Warren et al., 2007]

$$\mu(x,t) = \frac{p'(t) \times p''(t)}{\big((p(t)-x) \times p'(t)\big)^2}$$

- Mean value kernel [Dyken and Floater, 2009]

$$\mu(x,t) = \frac{(p(t)-x) \times p'(t)}{\|p(t)-x\|^3}$$

- Laplace kernel [Kosinka and Bartoň, 2016]

$$\mu(x,t) = \frac{2\|p'(t)\|^2 (p(t)-x) \times p'(t) - \|p(t)-x\|^2 p'(t) \times p''(t)}{\big((p(t)-x) \times p'(t)\big)^2}$$

Related work

- Wachspress kernel [Warren et al., 2007] *only for convex domains*

$$\mu(x,t) = \frac{p'(t) \times p''(t)}{\big((p(t)-x) \times p'(t)\big)^2}$$

- Mean value kernel [Dyken and Floater, 2009] *can be negative*

$$\mu(x,t) = \frac{(p(t)-x) \times p'(t)}{\|p(t)-x\|^3}$$

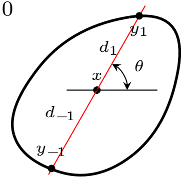
- Laplace kernel [Kosinka and Bartoň, 2016] *only for convex domains*

$$\mu(x,t) = \frac{2\|p'(t)\|^2 (p(t)-x) \times p'(t) - \|p(t)-x\|^2 p'(t) \times p''(t)}{\big((p(t)-x) \times p'(t)\big)^2}$$

Related work

- Gordon–Wixom interpolation [Gordon and Wixom, 1974]

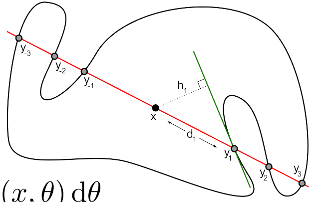
$$g_{1,-1}(x,\theta) = \left(\frac{f(y_1)}{d_1} + \frac{f(y_{-1})}{d_{-1}}\right) \Big/ \left(\frac{1}{d_1} + \frac{1}{d_{-1}}\right), \quad g(x) = \frac{1}{2\pi}\int_0^{2\pi} g_{1,-1}(x,\theta)\,\mathrm{d}\theta$$



- Weighted Gordon–Wixom interpolation [Belyaev, 2006]

$$g(x) = \int_0^{2\pi} g_{1,-1}(x,\theta)\omega(x,\theta)\,\mathrm{d}\theta \Big/ \int_0^{2\pi} \omega(x,\theta)\,\mathrm{d}\theta$$

- Positive Gordon–Wixom kernel [Manson et al., 2011]

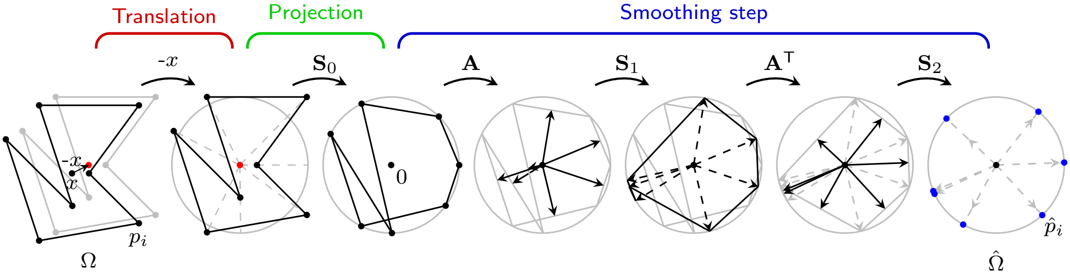


$$g_{i,j}(x,\theta) = \left(\frac{f(y_i)}{d_i} + \frac{f(y_{-j})}{d_{-j}}\right) \Big/ \left(\frac{1}{d_i} + \frac{1}{d_{-j}}\right)$$

$$g(x) = \int_0^{2\pi} \sum_{i=1}^{m} \sum_{j=-n}^{-1} g_{i,j}(x,\theta)\omega_{i,j}(x,\theta)\,\mathrm{d}\theta \Big/ \int_0^{2\pi} \sum_{i=1}^{m} \sum_{j=-n}^{-1} \omega_{i,j}(x,\theta)\,\mathrm{d}\theta$$

Discrete max-likelihood coordinates

[Chang et al., 2023]



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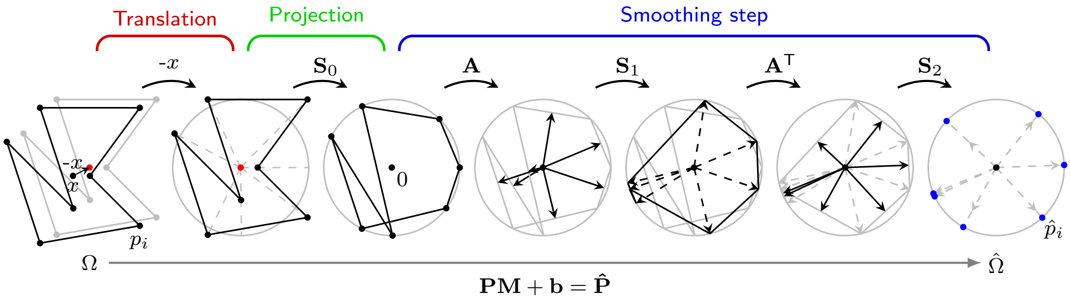

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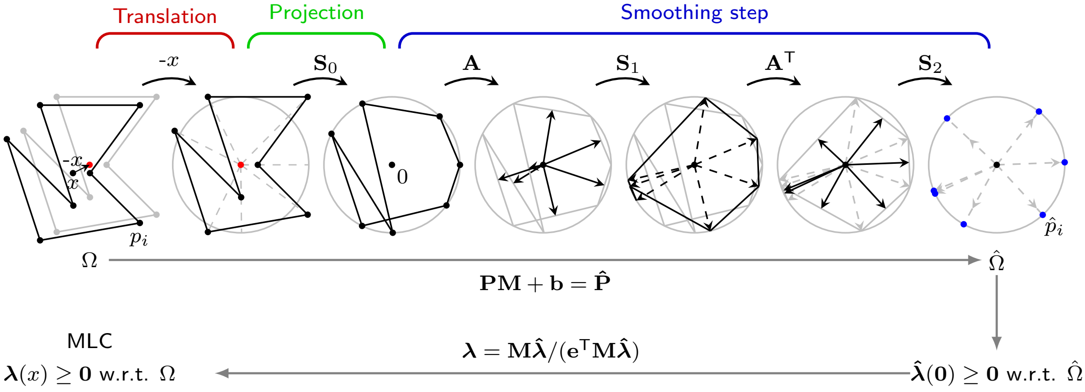

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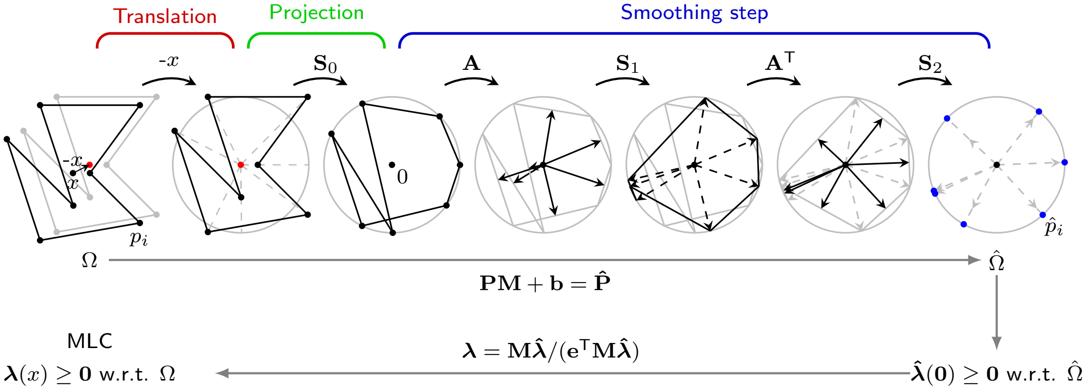
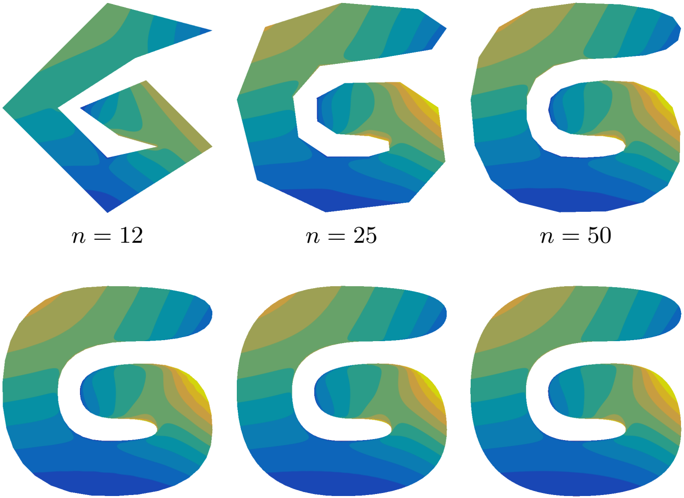


$$\hat{\boldsymbol{\lambda}} = \arg\max_{\hat{\boldsymbol{\lambda}}} \sum_{i=1}^{n} \log \hat{\lambda}_i \quad \text{s.t.} \quad \sum_{i=1}^{n} \hat{\lambda}_i \hat{p}_i = 0, \quad \sum_{i=1}^{n} \hat{\lambda}_i = 1$$

Discrete maximum likelihood interpolation



$n = 12$ $n = 25$ $n = 50$

$n = 100$ $n = 200$ $n = 500$

Extension to continuous boundaries

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- Translate and project the boundary onto the unit circle oriented at $\mathbf{0}$

$$\hat{p}(t) = \frac{p(t) - x}{r(x,t)}, \qquad r(x,t) = \|p(t) - x\|$$

- Smooth the adjacent points (not needed)

- Compute $\hat{\lambda}(t)$ by maximizing the likelihood function

$$\ell[\hat{\lambda}] = \int_a^b \log \hat{\lambda}(t)\,\mathrm{d}t \qquad \text{s.t.} \quad \int_a^b \hat{\lambda}(t)\hat{p}(t)\,\mathrm{d}t = \mathbf{0}, \qquad \int_a^b \hat{\lambda}(t)\,\mathrm{d}t = 1$$

- Compute the barycentric kernel $\lambda(x,t)$ over the original boundary

$$\lambda(x,t) = \frac{\hat{\lambda}(t)}{r(x,t)} \bigg/ \int_a^b \frac{\hat{\lambda}(s)}{r(x,s)}\,\mathrm{d}s$$

Extension to continuous boundaries

❖ Translate and project the boundary onto the unit circle oriented at $\mathbf{0}$

$$\hat{p}(t) = \frac{p(t) - x}{r(x,t)}, \qquad r(x,t) = \|p(t) - x\|$$

❖ Smooth the adjacent points (not needed)

Positive weight function

❖ Compute $\hat{\lambda}(t)$ by maximizing the likelihood function

$$\ell[\hat{\lambda}] = \int_a^b \log \hat{\lambda}(t) w(t) \, \mathrm{d}t \qquad \text{s.t.} \quad \int_a^b \hat{\lambda}(t)\hat{p}(t) w(t) \, \mathrm{d}t = \mathbf{0}, \quad \int_a^b \hat{\lambda}(t) w(t) \, \mathrm{d}t = 1$$

❖ Compute the barycentric kernel $\lambda(x,t)$ over the original boundary

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Extension to continuous boundaries

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☞ Problem.1: How to solve the optimization problem?

☞ Problem.2: How to choose the weight function?

❖ Compute $\hat{\lambda}(t)$ by maximizing the likelihood function

$$\ell[\hat{\lambda}] = \int_a^b \log \hat{\lambda}(t) w(t) \, \mathrm{d}t \qquad \text{s.t.} \quad \int_a^b \hat{\lambda}(t)\hat{p}(t) w(t) \, \mathrm{d}t = \mathbf{0}, \quad \int_a^b \hat{\lambda}(t) w(t) \, \mathrm{d}t = 1$$

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Solving the optimization problem

With the *Lagrange multipliers method* and the *Euler-Lagrange equation*

$$\hat{\lambda}(t) = \frac{1}{\phi_0 + \phi^{\mathsf{T}} \hat{p}(t)}, \quad \phi_0 = \int_a^b w(t)\, \mathrm{d}t$$

Solving the optimization problem

With the *Lagrange multipliers method* and the *Euler-Lagrange equation*

$$\hat{\lambda}(t) = \frac{1}{\phi_0 + \phi^{\mathsf{T}} \hat{p}(t)}, \quad \phi_0 = \int_a^b w(t)\,\mathrm{d}t$$

The ϕ can be computed by minimizing the function

$$\phi = \arg\min_{\phi \in \Phi} F(\phi), \quad F(\phi) = -\int_a^b \log(\phi_0 + \phi^{\mathsf{T}} \hat{p}(t)) w(t)\,\mathrm{d}t$$

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The domain of F is a circle with radius ϕ_0 centered at the origin

$$\Phi = \{\phi \in \mathbb{R}^2 \mid \|\phi\| \leq \phi_0\}$$

Solving the optimization problem

With the *Lagrange multipliers method* and the *Euler-Lagrange equation*

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The domain of F is a circle with radius ϕ_0 centered at the origin

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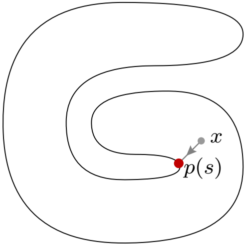
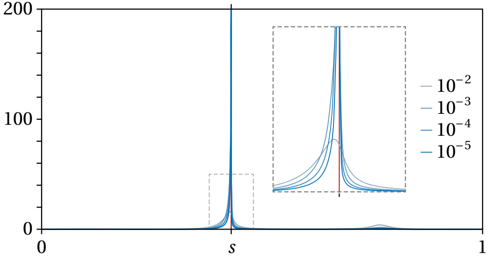
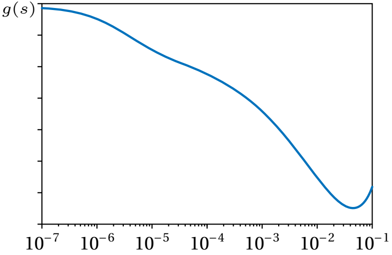
$F(\phi)$: strictly convex function　　Newton method

❖ Constant weight function

$$w_x(t) = 1$$





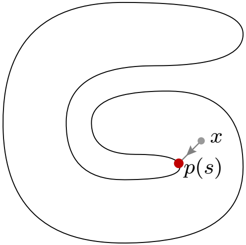


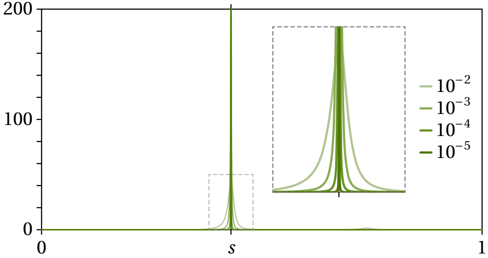
$\lambda(x, t)$ converges slowly to δ function

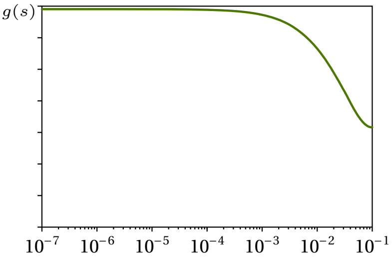
$g(x)$ converges slowly to $g(s)$

❖ Inverse distance weight function

$$w_x(t) = \frac{1}{r(x,t)}$$





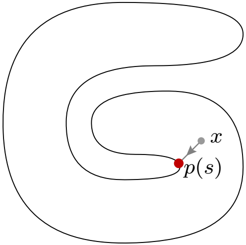


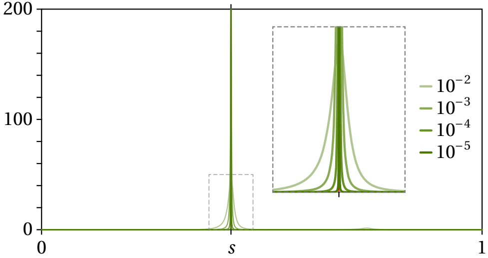
$\lambda(x,t)$ converges quickly to δ function

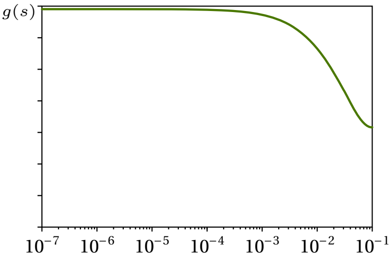
$g(x)$ converges quickly to $g(s)$

❖ Inverse distance weight function

$$w_x(t) = \frac{1}{r(x,t)}$$





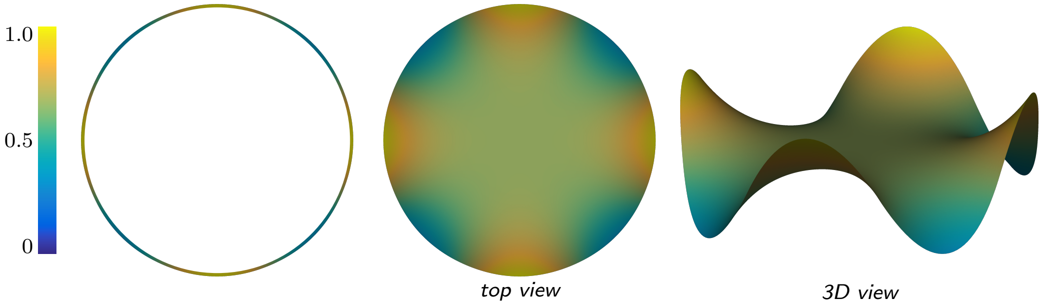


$\lambda(x,t)$ converges quickly to δ function

$g(x)$ converges quickly to $g(s)$

$\lambda(x,t)$ is a **pseudo-harmonic kernel**

❖ Inverse distance weight function



harmonic function

❖ k-th power of inverse distance weight function

$$w_x(t) = \frac{1}{r(x,t)^k}$$

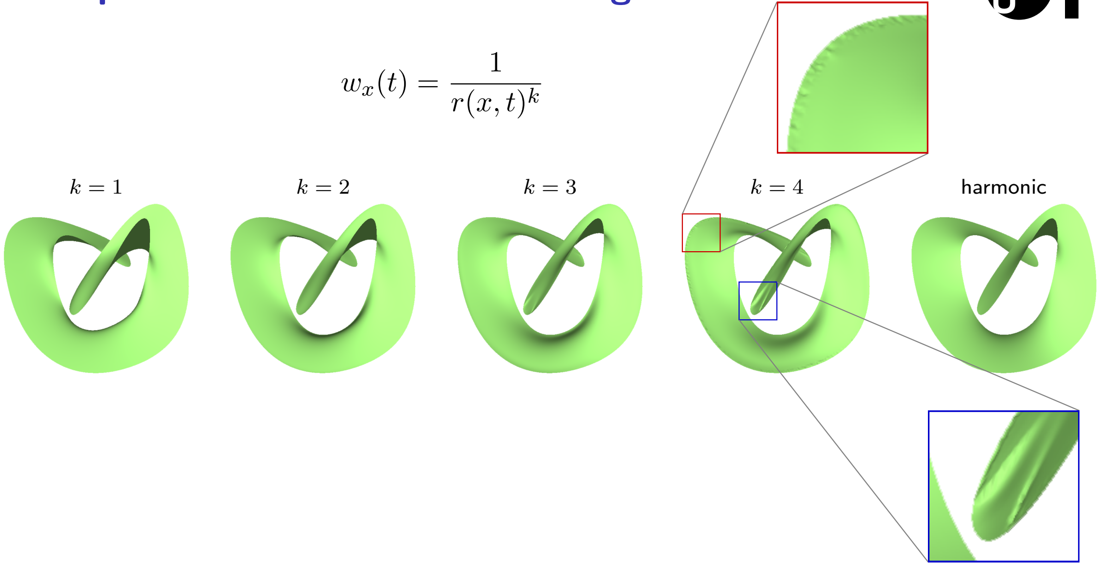
❖ k-th power of inverse distance weight function

$$w_x(t) = \frac{1}{r(x,t)^k}$$

$k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | harmonic

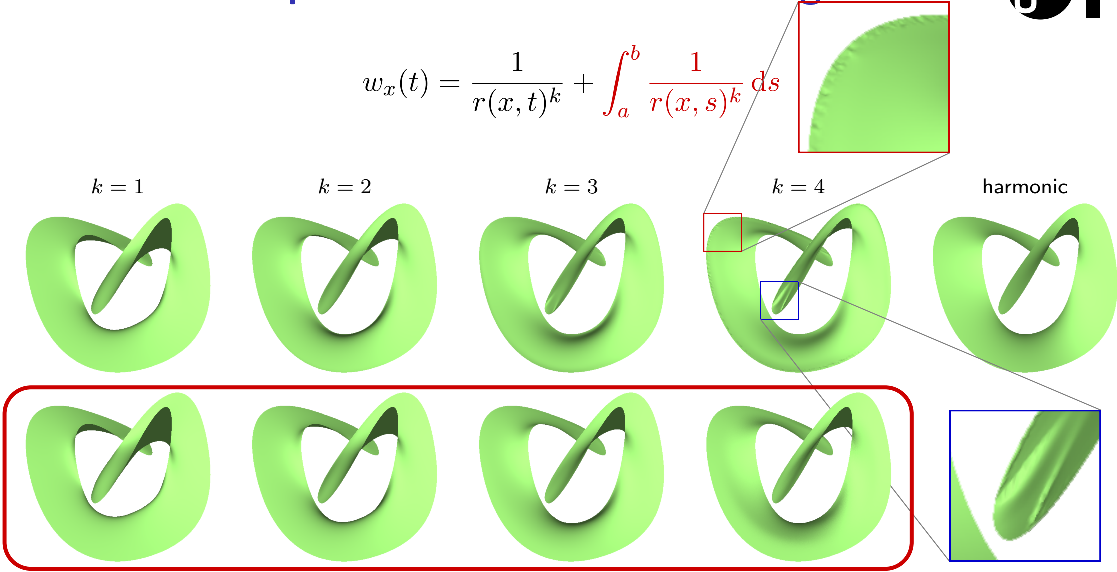
❖ k-th power of inverse distance weight function

$$w_x(t) = \frac{1}{r(x,t)^k}$$



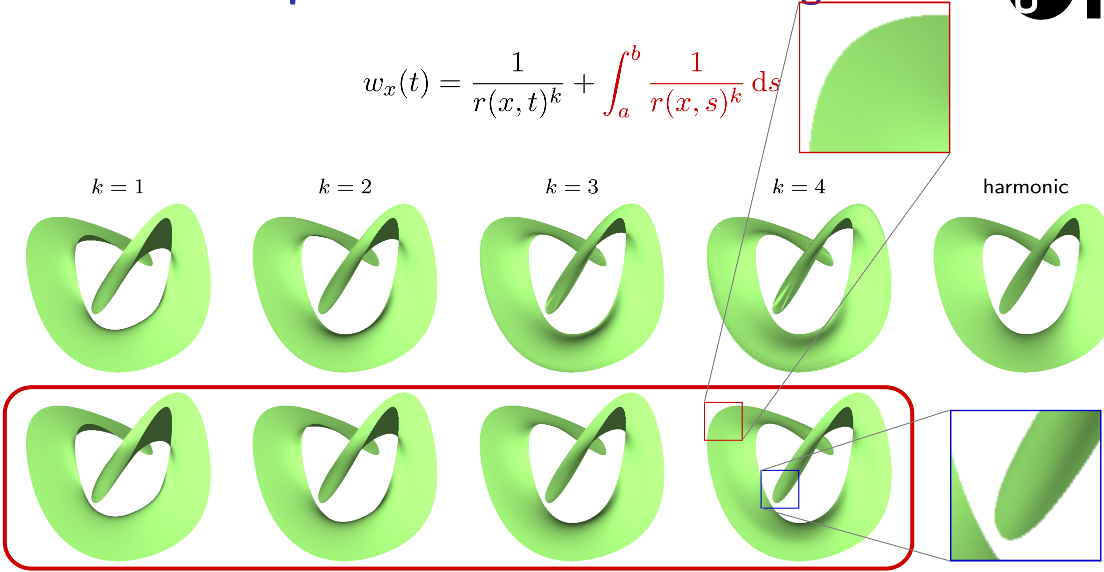
❖ Modified k-th power of inverse distance weight

$$w_x(t) = \frac{1}{r(x,t)^k} + \int_a^b \frac{1}{r(x,s)^k}\,ds$$



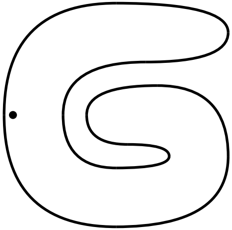
❖ Modified k-th power of inverse distance weight

$$w_x(t) = \frac{1}{r(x,t)^k} + \int_a^b \frac{1}{r(x,s)^k}\,\mathrm{d}s$$



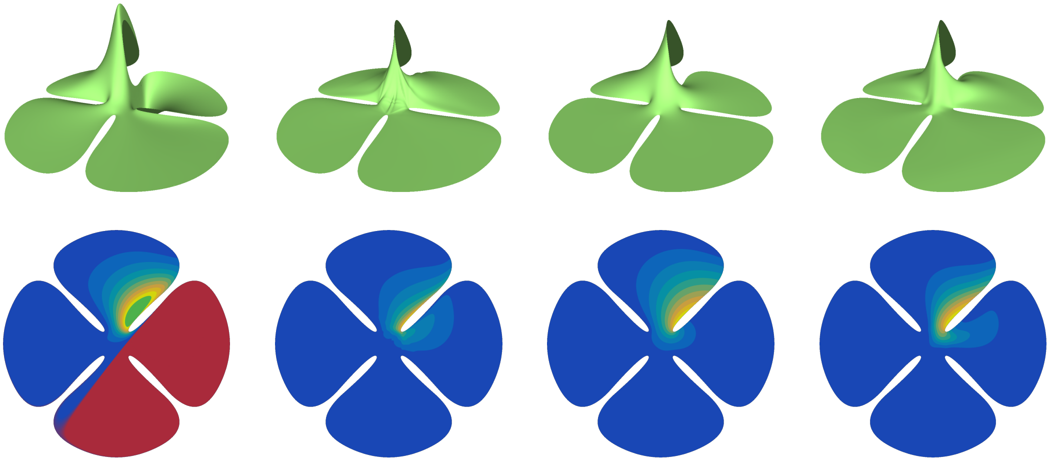
Reason for improving stability

$$w_x(t) = \frac{1}{r(x,t)^4} \qquad\qquad w_x(t) = \frac{1}{r(x,t)^4} + \int_a^b \frac{1}{r(x,s)^4}\,\mathrm{d}s$$



$$F(\phi) = -\int_a^b \log(\phi_0 + \phi^\mathsf{T}\hat{p}_x(t))w_x(t)\,\mathrm{d}t$$

Experiments - Comparison of interpolation



mean value kernel | positive Gordon–Wixom | Poisson kernel | $w_x(t) = \frac{1}{r^4} + \int_a^b \frac{1}{r^4}\,\mathrm{d}s$

Experiments - Interpolating polygonal boundary domain

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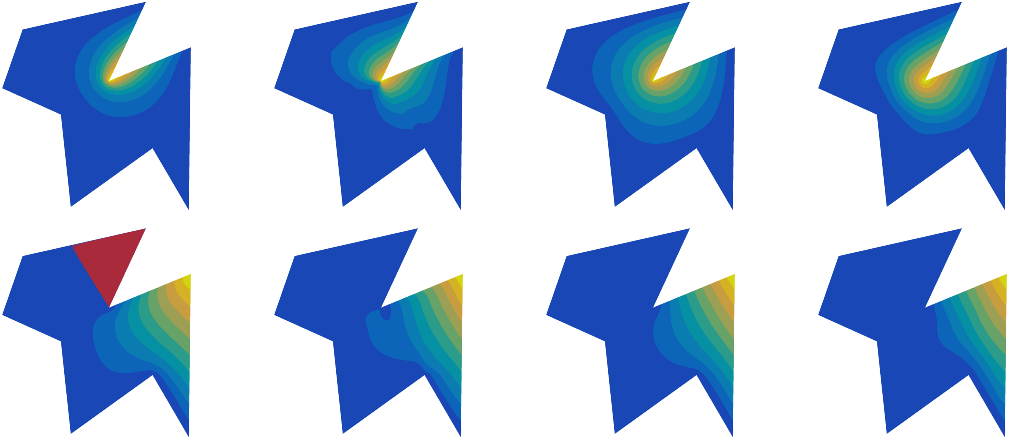
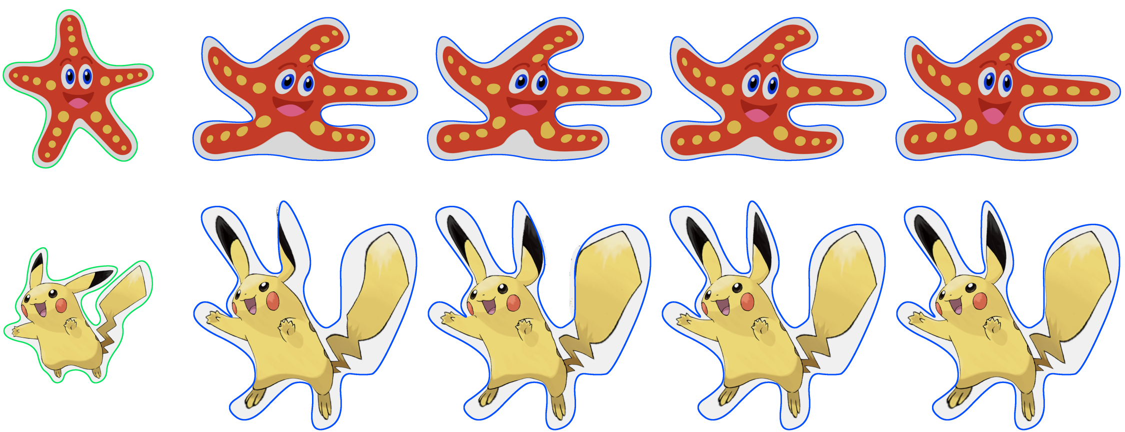
generalized barycentric coordinates



mean value kernel | positive Gordon–Wixom | Poisson kernel | $w_x(t) = \frac{1}{r^4} + \int_a^b \frac{1}{r^4}\,\mathrm{d}s$

Experiments - Comparison of image deformation



source | mean value kernel | positive Gordon–Wixom | Poisson kernel | $w_x(t) = \frac{1}{r^4} + \int_a^b \frac{1}{r^4}\,\mathrm{d}s$

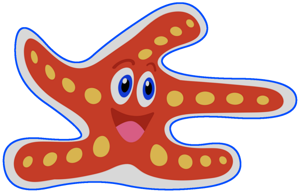
Conclusion

- Extend maximum likelihood coordinates to arbitrary closed domains
  - Variational problem *(Lagrange multipliers method; Euler-Lagrange equation)*
- By introducing a weight function, we get different results
  - Constant weight function *(equivalent to the discrete maximum likelihood coordinates)* *(convergence slowly to the δ function at the boundary.)*
  - Inverse distance weight *(pseudo-harmonic kernel)*
  - k-th power of inverse distance weight *(unstable optimization)*
  - Modified k-th power of inverse distance weight *(stable optimization)*

Positive & Smooth

- Parameterization-dependent
  - The natural parameterization is good enough for most cases
  - More freedom for the users

Thank you!



Running time

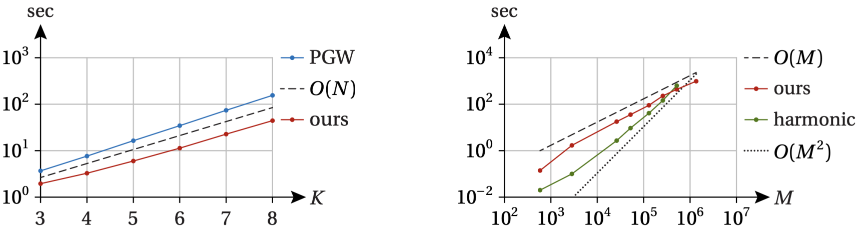


Figure 7: Running time in seconds for evaluating our interpolants and the PGW interpolant based on $N = 2^K 12$ boundary samples at 26000 domain points (left) and for computing the piecewise linear approximation of our interpolants and the harmonic interpolant over a domain triangulation with M vertices (right).

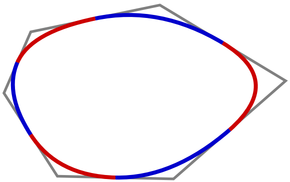
Experiments - Construct boundary and boundary data

Using quadratic B-splines for describing both the boundary and the boundary data

- Construct boundary

$$p(t) = \sum_{i=0}^{n+1} p_i B_i(t)$$

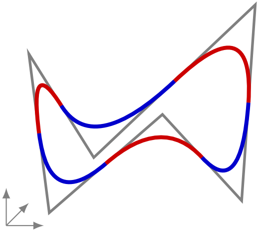
$$(p_0 := p_n,\ p_{n+1} := p_1)$$



- Specific boundary data

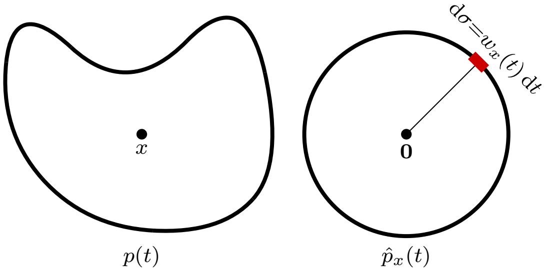
$$f(t) = \sum_{i=0}^{n+1} f_i B_i(t)$$

$$(f_0 := f_n,\ f_{n+1} := f_1)$$



❖ Arc length differential weight function

$$w_x(t) = \|\hat{p}_x'(t)\|$$



$$\arg\max \ell[b_x] = \oint_{\hat{p}_x} \log k_x \; d\sigma$$

s.t. $$\oint_{\hat{p}_x} k_x \; d\sigma = 1,$$

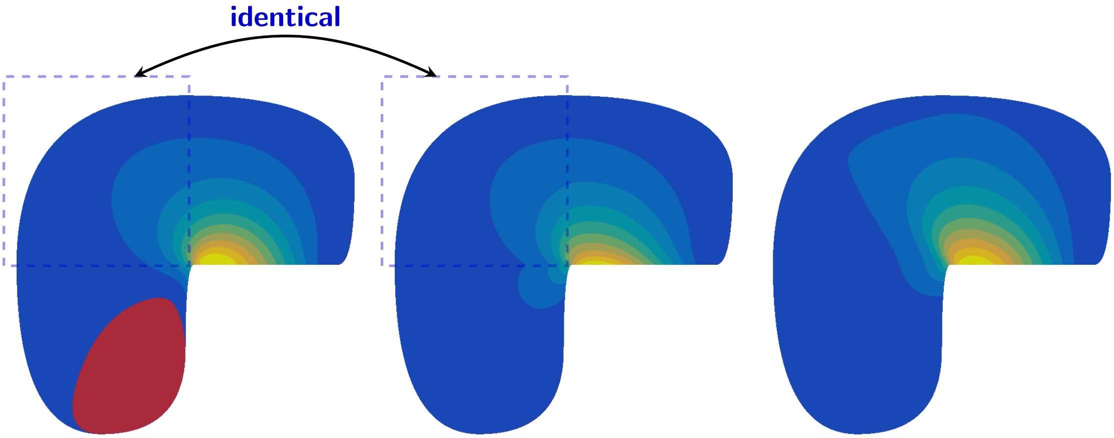
$$\oint_{\hat{p}_x} k_x \cdot \sigma \; d\sigma = \mathbf{0}.$$

C^0 continuity

$$w_x(t) = \frac{|(p(t) - x) \times p'(t)|}{r(x,t)^2} = |\mu(x,t)|r(x,t)$$

$\mu(x,t)$ is **mean value kernel**

❖ Arc length differential weight function



mean value kernel

$w_x(t) = \|\hat{p}'_x(t)\|$

$w_x(t) = \|\hat{p}'_x(t)\|^2$

How different parameterizations affect the results

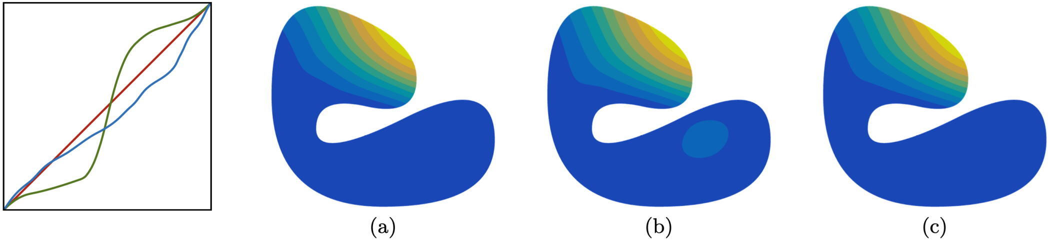


Figure 12: Transfinite basis function for our kernel with ω_x in (20) using the natural (red) parameterization of the boundary spline curve (a), after applying a monotonic (green) reparameterization (b), which creates a significant local maximum in the bottom right part of the domain, and after applying the arc-length (blue) reparameterization (c), which has little effect on the result in this example.

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— The PGW source code is available at
http://josiahmanson.com/research/gordon_wixom_coords/.

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