Università
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Maximum Likelihood Coordinates

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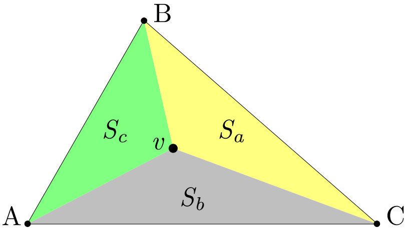
Barycentric coordinates in triangle

Given a triangle $\triangle \mathrm{ABC}$ with vertices $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and some $v \in \mathrm{Int}(\triangle \mathrm{ABC})$, find coordinates $\boldsymbol{\lambda} = [\lambda_a, \lambda_b, \lambda_c]^\mathsf{T}$ such that $v = \lambda_a \mathrm{A} + \lambda_b \mathrm{B} + \lambda_c \mathrm{C}$ and $\lambda_a + \lambda_b + \lambda_c = 1$.



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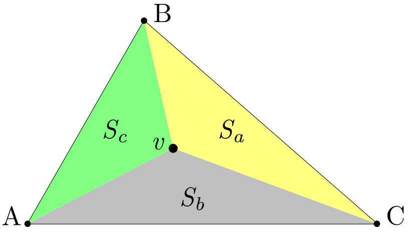
$\boldsymbol{\lambda}$ are barycentric coordinates of v with respect to $\triangle \mathrm{ABC}$.

Unique

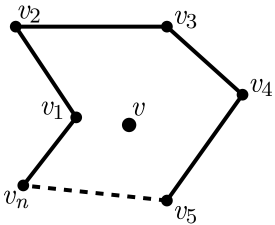
- $\boldsymbol{\lambda} = [\frac{S_a}{S}, \frac{S_b}{S}, \frac{S_c}{S}]^\mathsf{T}$
- $S = S_a + S_b + S_c$



Barycentric coordinates in polygon

Given a polygon Ω with n vertices $v_1, v_2, \ldots, v_n$ and any $v \in \Omega$, find coordinates $\boldsymbol{\lambda}(v) = [\lambda_1(v), \lambda_2(v), \ldots, \lambda_n(v)]^\mathsf{T}$ such that $v = \sum_i \lambda_i v_i$ and $\sum_i \lambda_i = 1$.

$\boldsymbol{\lambda}$ are generalized barycentric co-ordinates of v with respect to the polygon Ω.



- When the $n > 3$, such $\boldsymbol{\lambda}$ are usually **not unique**.
- Looking forward to finding $\boldsymbol{\lambda}$ that satisfy some properties.

Some properties we desire

Inherent properties:

- Reproduction: $v = \sum_{i=1}^{n} \lambda_i(v) v_i$
- Partition of unity: $\sum_{i=1}^{n} \lambda_i(v) = 1$

Some properties we desire

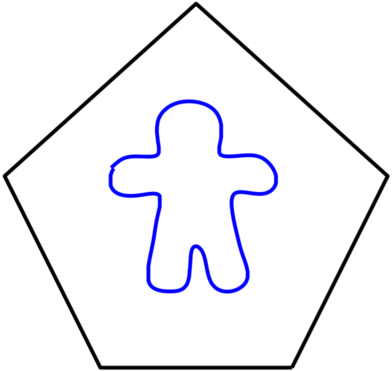
Inherent properties:

- Reproduction: $v = \sum_{i=1}^{n} \lambda_i(v) v_i$
- Partition of unity: $\sum_{i=1}^{n} \lambda_i(v) = 1$

More properties we desire:

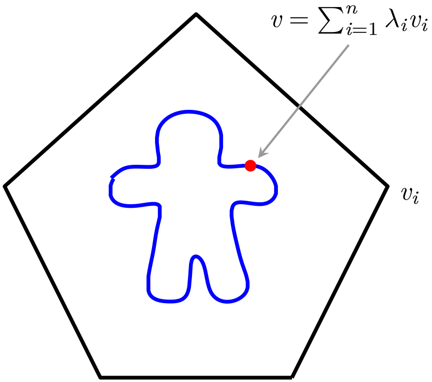
- Lagrange property: $\lambda_i(v_j) = \delta_{i,j}$
- Non-negativity: $\lambda_i(v) \geq 0$
- Piecewise linearity on $\partial\Omega$
- Smoothness: $\lambda_i(v) \in \mathrm{C}^k,\ v \in \Omega,\ k > 0$
- Closed form: there exists a simple formula to express and compute the weights $\boldsymbol{\lambda}(v)$

Free-Form deformation



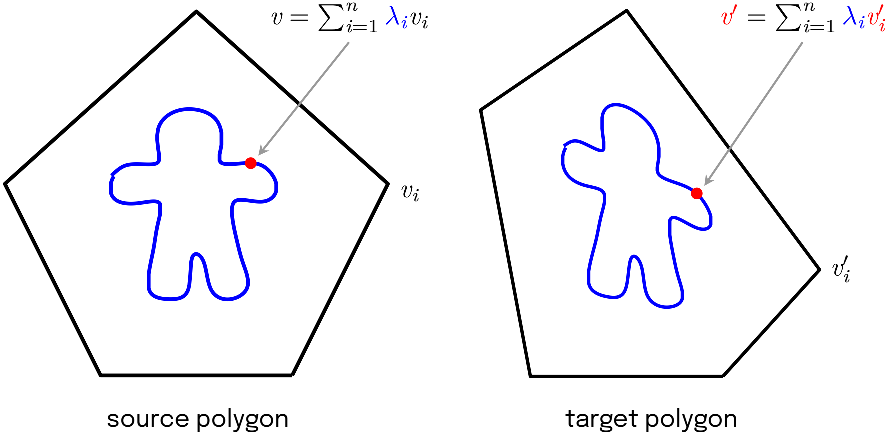
source polygon

Free-Form deformation



source polygon

Free-Form deformation

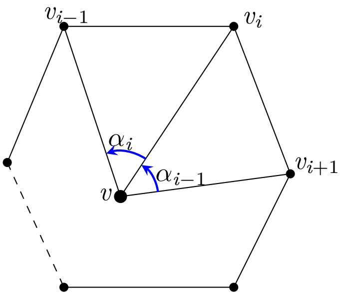


Related work

Mean value coordinates [Floater 2003]

$$\lambda_i(v) = w_i(v) / \sum_j w_j(v)$$

$$w_i(v) = \frac{\tan\frac{\alpha_{i-1}}{2} + \tan\frac{\alpha_i}{2}}{\|v_i - v\|}$$



- Note: α_i is a **signed angle**.

- **Advantage:** closed form; smooth
- **Disadvantage:** may take negative values in some concave polygons

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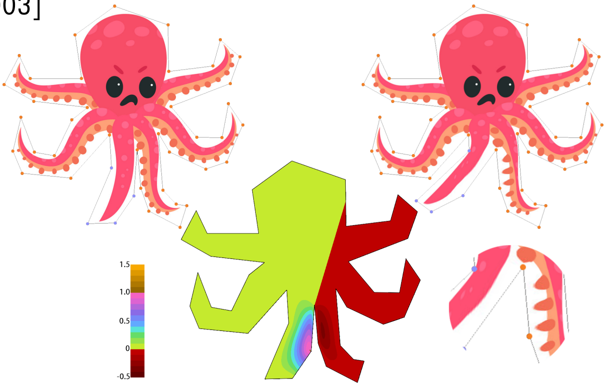

ref. [Deng et al., 2020, CAGD]

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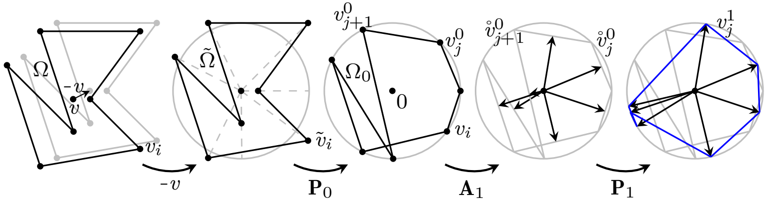
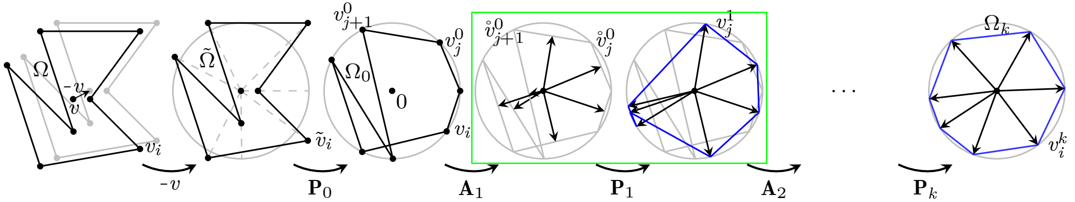
Related work

Iterative coordinates [Deng et al. 2020]



Related work

Iterative coordinates [Deng et al. 2020]



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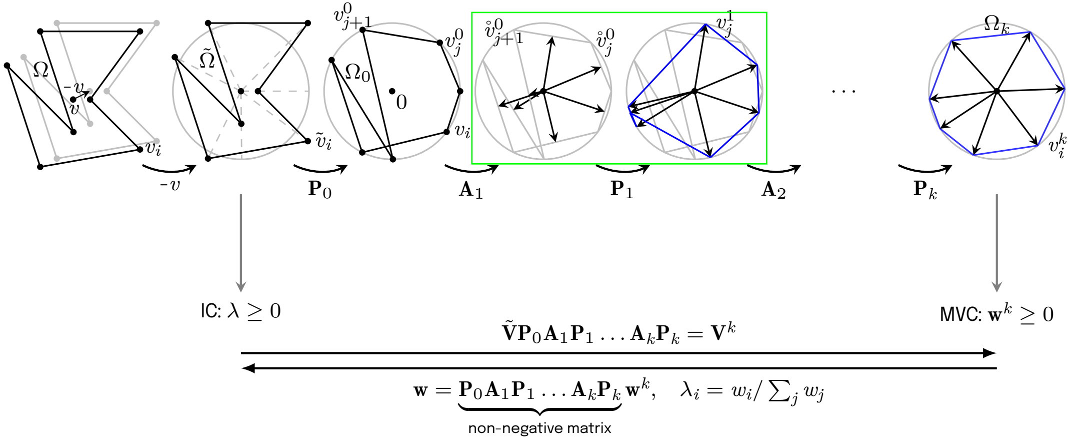
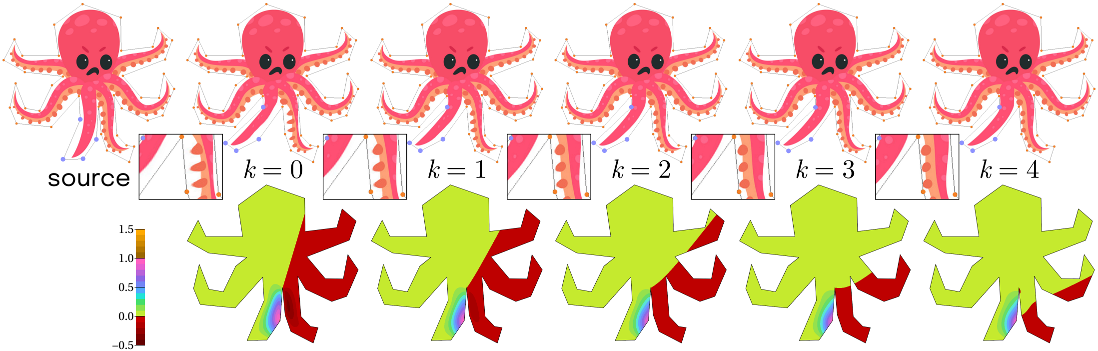


Related work



ref. [Deng et al., 2020, CAGD]

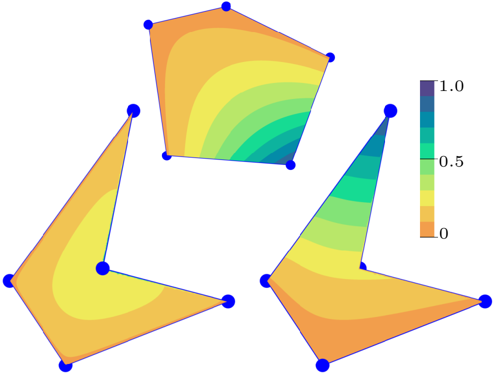
Related work

Maximum entropy approach [Sukumar 2004]

$$\max_{\lambda(v)\in\mathbb{R}^n_+} H(\lambda)$$

$$H(\lambda) = -\sum_{i=1}^{n} \lambda_i(v) \log \lambda_i(v)$$

s.t. $\quad v = \sum_{i=1}^{n} \lambda_i(v) v_i \qquad \sum_{i=1}^{n} \lambda_i(v) = 1$



- It works well for convex polygons

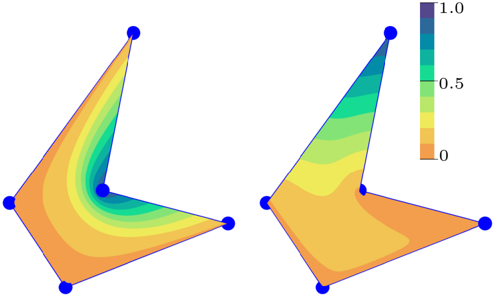
Related work

Maximum entropy coordinates [Hormann & Sukumar 2008]

$$\max_{\lambda(v)\in\mathbb{R}^n_+} H(\lambda, m)$$

$$H(\lambda, m) = -\sum_{i=1}^{n} \lambda_i(v) \log \frac{\lambda_i(v)}{m_i(v)}$$

s.t. $\quad v = \sum_{i=1}^{n} \lambda_i(v) v_i \qquad \sum_{i=1}^{n} \lambda_i(v) = 1$



- Note: A suitable set of **prior functions** m are required.

Basic maximum likelihood coordinates

For any $v \in \text{Int}(\Omega)$, we define the barycentric coordinates $\boldsymbol{\lambda} = \boldsymbol{\lambda}(v) \in \mathbb{R}^n$ by maximizing

$$\mathcal{L}(\boldsymbol{\lambda}) = \prod_{i=1}^{n} \lambda_i$$

subject to the constraints

$$\sum_{i=1}^{n} \lambda_i = 1, \qquad \sum_{i=1}^{n} \lambda_i v_i = v, \qquad \boldsymbol{\lambda} \in \mathbb{R}_+^n$$

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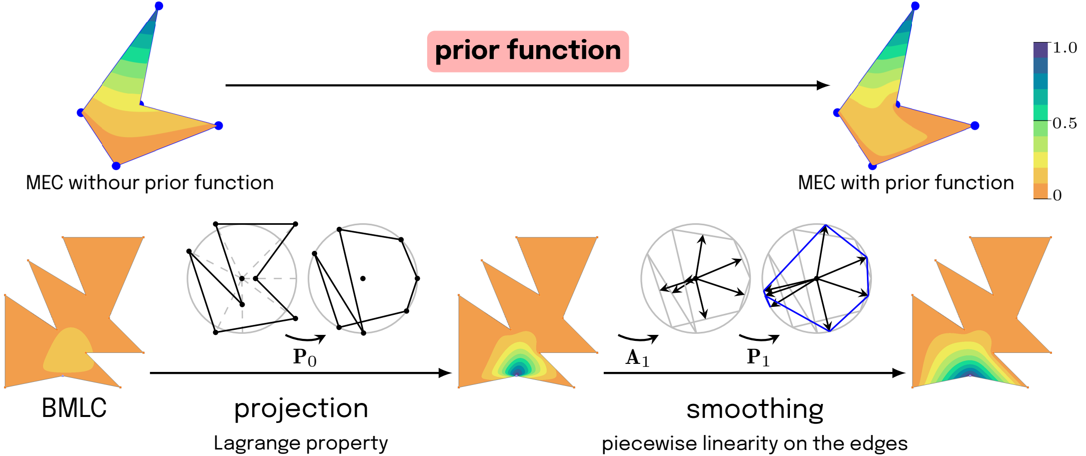
with the method of Lagrange multipliers:

$$\lambda_i = \frac{1}{n + \phi^\mathsf{T}(v_i - v)} \qquad \phi = \min_{\phi \in \mathbb{R}^2} F(\phi), \quad F(\phi) = -\sum_{i=1}^{n} \log(n + \phi^\mathsf{T}(v_i - v))$$

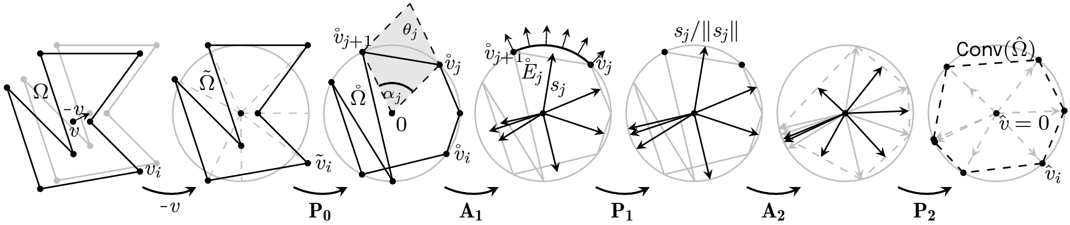
Basic maximum likelihood coordinates

- non-negative coordinates
- smoothness
- Lagrange property (Convex polygons)
- piecewise linearity on the edges (Convex polygons and convex edges of concave polygons)

Extension for non-convex polygons



Maximum likelihood coordinates



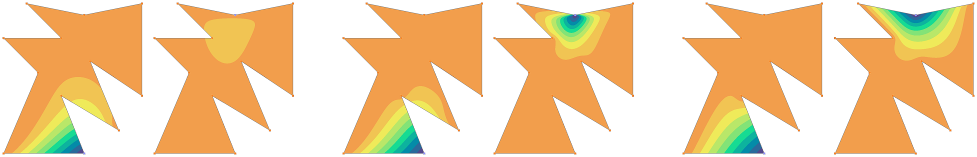
MLC: $\lambda \geq 0$

Basic MLC: $\hat{\mathbf{w}} \geq 0$

$$\tilde{\mathbf{V}}\mathbf{P}_0\mathbf{A}_1\mathbf{P}_1\mathbf{A}_2\mathbf{P}_2 = \hat{\mathbf{V}}$$

$$\mathbf{w} = \underbrace{\mathbf{P}_0\mathbf{A}_1\mathbf{P}_1\mathbf{A}_2\mathbf{P}_2}_{\text{non-negative matrix}} \hat{\mathbf{w}}, \quad \lambda_i = w_i / \textstyle\sum_j w_j$$

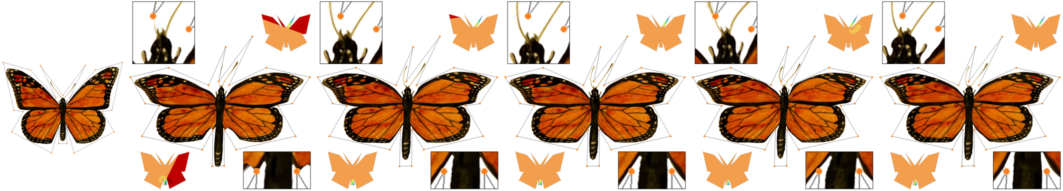
Maximum likelihood coordinates



(basic MLC) (with projection) (with projection and smoothing)

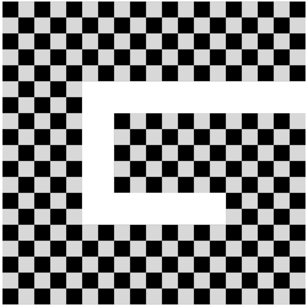
- non-negative coordinates
- smooth
- Lagrange property
- piecewise linearity on the edges

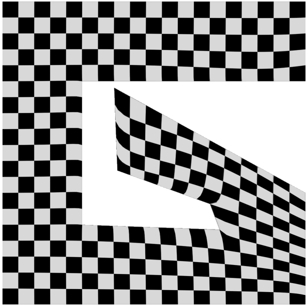
Deformation



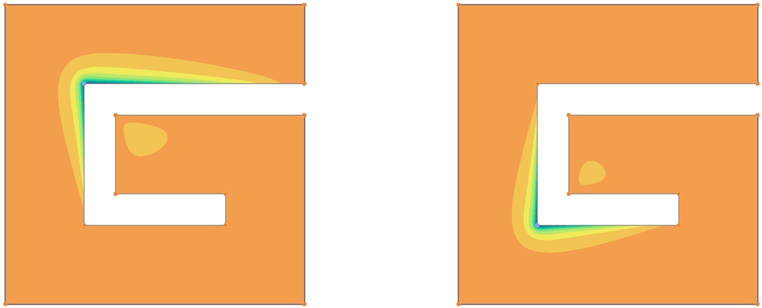
source | MVC [Floater 2003] | IC($k=4$) [Deng et al. 2020] | **MLC** | MEC [Hormann et al. 2008] | HC [Joshi et al. 2007]

Deformation





Scaled maximum likelihood coordinates



Scaled maximum likelihood coordinates

$$\mathbf{D} = diag\{\frac{1}{d_1}, \ldots, \frac{1}{d_n}\}$$

MLC: $\lambda \geq 0$

Basic MLC: $\mathbf{w}' \geq 0$

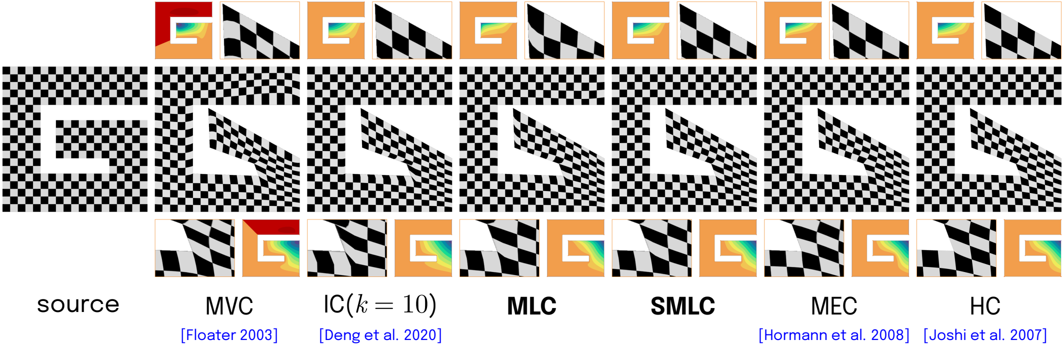
$$\tilde{\mathbf{V}}\mathbf{P}_0\mathbf{A}_1\mathbf{P}_1\mathbf{A}_2\mathbf{P}_2\mathbf{D} = \mathbf{V}'$$

$$\mathbf{w} = \underbrace{\mathbf{P}_0\mathbf{A}_1\mathbf{P}_1\mathbf{A}_2\mathbf{P}_2\mathbf{D}}_{\text{non-negative matrix}}\,\mathbf{w}', \quad \lambda_i = w_i / \sum_j w_j$$

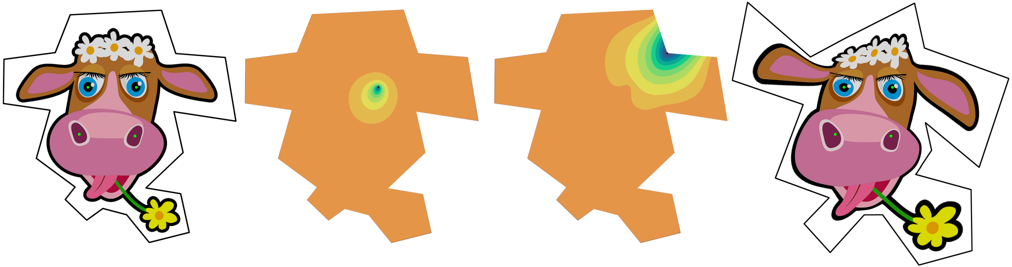
Scaled maximum likelihood coordinates

MLC | MLC with scaling | MLC | MLC with scaling

Deformation



Deformation with interior points

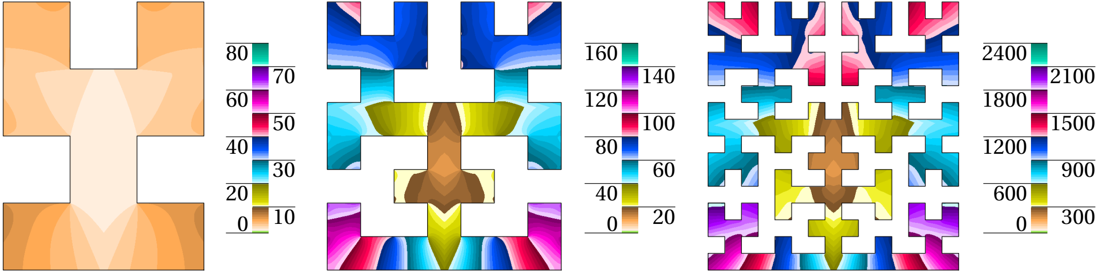


Conclusion

- IC[Deng et al. 2020]: it is difficult to determine the number of iterations
- MEC[Hormann & Sukumar 2008]: it is difficult to determine the prior function
- Maximum likelihood coordinates:
    - instead of entropy, we maximize likelihood (similar but simpler)
    - instead of the prior functions, we use iteration from iterative coordinates.
        - Lagrange property
        - piecewise linearity on the edges
    - non-negative
    - smooth
    - local maxima can be reduced by introducing an additional scaling step
    - gradient can be easily evaluated (chain rule)

Thank you!

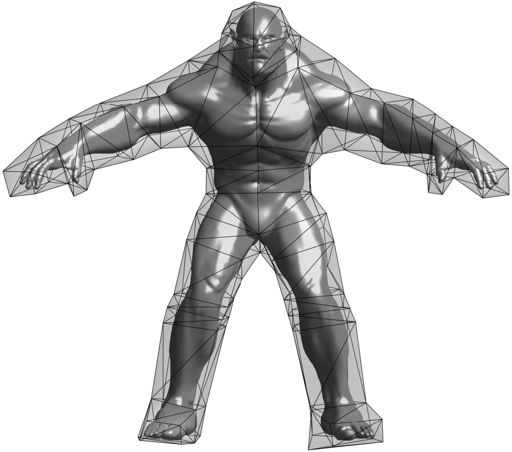
Appendix

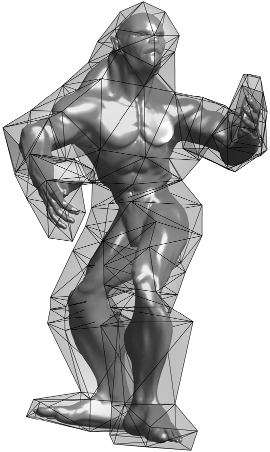


$(n, k_{\min}) = (18, 7)$ $(66, 128)$ $(258, 2139)$

Number of iterations required to get positive coordinates at the individual interior points of the (closed) Hilbert curves H_2, H_3, H_4.

Appendix





Appendix